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Nonlinear flight dynamics analysis

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THE ROYAL SOCIETY

Nonlinear flight dynamics analysis

By F. B. J. Macmillen

*Flight Sciences Department, Bombardier Aerospace Inc.,
400 Chemin de la Côte Vertu, Montréal, Quebec, Canada H4S 1Y9*

This introduction has two aims. First, it begins the issue with an overview of how modern nonlinear-dynamics analysis techniques may be of benefit to the flight dynamicist. Each paper that follows highlights a particular aspect of this application: a brief outline and discussion of each paper is presented. Second, the introduction aims to improve the accessibility of the theme issue to both flight dynamicists and nonlinear dynamicists unfamiliar with each other's fields. A brief introduction to both fields is presented, terminology explained, and references to 'classic' and recent introductory books provided.

Keywords: bifurcation analysis; terminology of nonlinear dynamics; flight dynamics, background; flight dynamics analysis

1. Background and aims of this theme issue

In the past few decades computing capability has enabled the analysis of *nonlinear* aircraft models and dynamics. For the most part, this analysis has consisted of nonlinear simulation, either through 'engineering' simulation tools or through 'pilot-in-the-loop' simulation. Capabilities, particularly in the latter, remain an active area of research and development. Less well known within the flight-dynamics (stability and control) community is the development of additional nonlinear analysis tools and concepts of benefit to aircraft dynamic analysis. Nonlinear dynamics as a discipline in its own right has resulted in two key developments: a terminological framework which describes, and aids the understanding of, nonlinear dynamic behaviour; and the development of tools to analyse nonlinear systems.

The reason behind this theme issue is that these developments can be of direct benefit to flight dynamicists in industry and academia. This paper aims to explain, illustrate and discuss this 'new' approach to flight dynamics analysis.

Within the flight dynamics community, nonlinearity has not been ignored; emphasis has been placed upon specific problem areas such as estimating or modelling nonlinear aerodynamics and the effects of specific nonlinearities on aircraft with active control systems. The modelling issue is, itself, a significant and problematic area of research and it is not discussed directly here.

The approaches presented in this issue are not brand new. The first application of bifurcation analysis (one of the new techniques) to aircraft dynamics was in 1977 (Mehra *et al*. 1977), at a time when the technique was limited by the computing power available. Since 1977, many studies (of the order of at least 50–60 published articles) from many different countries have applied the new analysis techniques to aircraft dynamics. The technique has not, however, been adopted as an engineering tool in the industry. The reasons for this are not certain, but several contributing reasons are clear.

1. The technique was limited, when first applied, by the high cost and limited power of computing at that time.

2. The new techniques developed have come 'wrapped-up' in the terminology of modern nonlinear dynamics, which is usually unfamiliar to the engineer in industry.

3. The analysis technique is only as good as the model: if the 'weak' point in the design process is the model, industry will put effort into improving models rather than improving analysis.

4. The new analysis techniques do (of course!) have their own limitations: as shown in the papers which follow.

Nevertheless, interest in these new approaches to analysis has continued and the techniques have been developed in the last two decades. There is some indication of change: one of the latest textbooks on aircraft stability and control (Abzug & Larrabee 1997) recognizes, for the first time, bifurcation analysis as a tool for the flight dynamicist. This book, which presents a historical review of flight dynamics, also points out that an underlying theme of that history has been the lag between theory and practice.

Modern nonlinear dynamics recognizes four main aspects of analysis:

1. derivation of trim (equilibrium) solutions (initial solutions and continuation schemes);

2. location and identification of bifurcations (bifurcation analysis);

3. estimation of basins of attraction; and

4. analysis of transient motion (simulation).

Together, these four aspects of analysis give as complete a picture of a nonlinear dynamical system as current understanding allows. Current practice in the aircraft industry, however, centres upon the last technique alone. The aim of this paper is to draw together in one publication, examples of the application of all four of the above techniques to flight dynamics. By so doing, it is hoped that awareness will be heightened of nonlinear dynamics of aircraft and the analysis tools available. In addition to presenting results, the papers presented describe the limitations and developments required in the techniques.

Each paper highlights a different aspect, application of, or approach to modern analysis of nonlinear flight dynamics.

The first paper is by Guicheteau and presents the broad range of work performed at ONERA, France. This paper is probably unique in providing an example of flight-test results. However, as the author points out, having obtained agreement between bifurcation analysis and simulation, correlation with flight test 'merely' shows that one's aircraft model is realistic. In addition, the point is made that in flight-test a pilot may not wish to, or be able to, sustain a given control configuration for a sufficient length of time for the steady-state dynamics to become clear. Guicheteau's paper presents further areas of research interest at ONERA: application to control law design; calculation of basins of attraction; effects of transient behaviour; and analysis of mixed continuous/discrete time systems.

The second paper is by Patel & Littleboy. It provides a clear example of how modern techniques (namely bifurcation analysis) may provide a 'map' of the aircraft's global dynamics: a map which may even be used by test pilots to enable them to fly to specific flight regimes (spin modes, etc.) and recover in a controlled and 'educated' manner. This paper is, perhaps, unique in showing how the techniques may aid the understanding of an aircraft's dynamics by both pilots and engineers.

In the paper by Jahnke, bifurcation analysis is used to aid an investigation of roll-coupling instabilities that result in jump phenomena. By comparing continuation diagrams, which show where the jump occurs, with moment-balance diagrams, the key factors which trigger the jump are discerned.

The paper by Lowenberg & Champneys considers an example of roll auto-rotation and subsequent dynamic behaviour of an F-4 combat-aircraft model. This paper is unique in providing an example of a global bifurcation, and its implications, in the analysis of flight dynamics. Global bifurcations are often not identified directly by continuation/bifurcation algorithms. In this particular case the impending homoclinic bifurcation is discerned by observing that the period of the limit-cycle solution nears infinity as the bifurcation point is approached.

In the paper by Liebst, bifurcation analysis is used to examine the onset of wing rock on an F-15, in order to develop a simple prediction method for onset of wing rock. His paper illustrates the simple techniques sought by the aircraft industry, and raises the issue of how important the global (and complex) picture presented by modern techniques is to industry. The approach developed in this paper requires further validation but is attractive in its simplicity. As the author points out, a more global approach, which uses modern techniques directly, will only be accepted if suitable software is developed.

The paper by Goman & Khramtsovsky applies modern nonlinear dynamics analysis to designing control laws for spin recovery and for wing-rock suppression. The use of these tools within a design cycle (as opposed to a flight-clearance analysis) is a more complex, and less developed, application. The approach adopted in this paper for wing-rock analysis and suppression may be contrasted with that in the paper by Liebst. This paper aims at capturing the full nonlinear and global dynamics, and the effects of the feedback schemes on these dynamics. The work by Goman & Khramtsovsky perhaps uses some of the most suitable algorithms and software yet developed for applications to aircraft dynamics. Their software uses a robust continuation scheme (which can cope with discontinuities in aerodynamic tables and control systems), employs a systematic method for finding solution branches and uses a technique for estimating basins of attraction, perhaps the most difficult of the four analysis steps listed above.

The paper by Lowenberg also considers the use of modern analysis techniques within the design problem, and provides perhaps the most extreme and far-reaching application of modern techniques presented in this theme issue. He considers how a designer may (radically) change the bifurcation behaviour of an aircraft to avoid certain undesirable bifurcations. This approach may be controversial: usually it would be preferable to suppress a bifurcation altogether rather than create a new one in order to avoid the original problem. An important point is made however: by considering the global dynamics, there may be an attractor 'out there' with a certain basin

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of attraction which could help the designer achieve a certain dynamic behaviour. Finally, Lowenberg raises another important issue: just how extreme will the flow regime be in the next generation of combat aircraft? Research at high angles of attack is very active, but remains highly debatable within the military aircraft industry.

The paper by Macmillen & Thompson aims to identify the benefits and limitations of the new techniques when applied to an industrial-scale aircraft model based upon the Harrier II aircraft. Some sample results are presented and the limitations of the software used is noted. The authors consider the benefits of modern nonlinear analysis techniques to industry and present a specification of requirements for a software package that would meet the needs of industry. In agreement with the conclusion of Professor Liebst, the authors believe that the development of suitable software is crucial to the techniques being adopted by industry.

Due to limitations of space and time, the number of contributions to this theme issue has been limited, some recent papers not included, but which deserve particular mention, are Avanzin & de Matteis (1996) and Planeaux *et al*. (1990). Both these papers provide good examples of the application of bifurcation analysis to aircraft with control augmentation systems. A paper by Mehra & Prasanth (1998) is also notable for applying continuation/bifurcation analysis to the problem of pilot-induced oscillations (PIOs), also referred to as aircraft–pilot coupling (APC). PIOs are undesirable oscillations resulting from the pilot's and aircraft's combined dynamics; often due to system nonlinearities, they are responsible for many modern combat aircraft incidents and accidents (McCruer *et al*. 1997). Mehra & Prasanth (1998) use a combined pilot–aircraft model and bifurcation analysis to search for Hopf bifurcations indicating the onset of PIO.

2. An introduction to nonlinear dynamics

For a thorough introduction to modern nonlinear dynamics, there are many good textbooks. For the engineer in industry, the book by Strogatz (1994) is an ideal introduction, Thompson & Stewart (1986) includes a good introduction with example case studies. Further case studies and a glossary of terminology is found in Thompson & Bishop (1993). The aim of this section is to explain some key ideas and terminology of nonlinear dynamics. The discipline has grown out of the pioneering work of Poincaré at the end of the 19th century. The developments since then are listed below.

1. An increased awareness of the importance and effects of nonlinearities in a system. The most publicized aspect of this has been the public interest in 'chaos'. Within the flight-dynamics community, examples would include an increased awareness of the importance of nonlinear terms in aerodynamic models (for stall modelling for example), or rate and amplitude limiting of actuators with an active flight control system.

2. Development of topological concepts to describe nonlinear dynamics. Topology is a technique of using geometry to describe a dynamical system (normally described by differential equations). Much of the terminology described below ('basins of attraction', 'attractors', 'saddles', 'insets and outsets') provides a visual way of describing dynamics.

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3. Developments in the theory of nonlinear differential equations (such as the Hartmann–Grobman theorem, 'implicit function' theorem and 'centre manifold' theorem). These are the principle mathematical pillars upon which the computer algorithms used are based.

4. Development of theories of stability: including ideas of 'structural stability' and 'basins of attraction'.

5. Developments in computational power and numerical schemes to implement the analysis techniques derived analytically.

Because all 'real-life' systems are nonlinear, it may be easy to forget the phenomena that exist purely due to nonlinearities.

1. Bounded but non-stationary behaviour, such as limit-cycle oscillations, is only possible in nonlinear systems. (The only exception being idealized structurally unstable linear models.)

2. The existence of more than one stable steady-state solution (trim condition) for a given set of system parameters is only possible with a nonlinear system. Hence, phenomena such as hysteresis and jumps between solutions are only possible in a nonlinear system.

3. The generation of complex (quasi-periodic or chaotic) transient or steady-state behaviour can only arise in a nonlinear system. Such behaviour signifies important consequences such as sensitivity to initial conditions.

In order to describe nonlinear dynamics, a vast array of terminology has been developed. The following is a brief glossary of key concepts used.

(*a*) *Phase space*

A phase space is usually referred to as 'state space' in the flight-dynamics community. Phase space is an imaginary 'space' in which each 'axis' represents the value of each phase variable (state variable). An aircraft is usually described by eight state variables: p, q, r, V, α, β, θ and ϕ (defined in § 3). The behaviour, or 'state', of an aircraft at any point in time with controls fixed may be described completely by a point in an eight-dimensional phase space. If the aircraft is not rigid or has an active control system, more states will be required to uniquely define the aircraft's 'state'.

If the aircraft is in a steady state (not changing with time), the aircraft's behaviour would be described by a single point in state space, since all the state variables will be maintaining a constant value. If the aircraft is perturbed in some way, transient motion will result in a trajectory being traced out in state space: eventually the trajectory will once more settle down into a 'steady state'. Such trajectories may be called the 'flow' of the system. The complete set of possible trajectories in phase/state space forms the system's 'phase portrait'. For any system, the trajectories in the complete phase space can never cross. For a system with more than two or three state variables only a two- or three-dimensional 'phase projection' may be viewed at a time.

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(b) *Phase-control space*

An aircraft also has certain controls: these may be viewed either as aerodynamic control surfaces (elevator, aileron, rudder, etc.) or control inceptors (the pilot's stick, rudder pedals, etc.). In either case the system may be described completely in phase-control space. For example, the rigid aircraft with eight state variables may have three controls (η, ξ, ζ) giving an 11-dimensional phase-control space. As the controls are varied, a trajectory in phase-control space is traced out. If the controls are changed slowly compared to the response of the aircraft, then the behaviour may be treated as 'quasi-stationary' and we have a true dynamical flow governed by a stationary vector field in just the phase space. Aircraft models with aeroelastics or flight-control systems modelled may contain many more states (or phase-space variables). The concepts which follow are generally illustrated in two dimensions, but are applicable in n dimensions.

(c) *Recurrent (steady-state) behaviour*

Engineers are familiar with the distinction between transient and steady-state behaviour. However, a nonlinear system may exhibit long-term behaviour which never settles to a 'steady' condition. An example is an electrical oscillator in a signal generator: the aim of such a system may be for the 'steady-state' behaviour to be a sinusoid; plotted in phase space (say, potential difference versus current), the 'steady-state' trajectory will be a closed loop (limit cycle), not a point. Is this steady-state behaviour? This issue also arises with quasi-periodic and chaotic 'steady-state' behaviour which nonlinear systems may exhibit. How can 'steady-state' be distinguished from transient behaviour? The most general definition of recurrent behaviour is called 'non-wandering' behaviour, in which arbitrarily close initial conditions eventually return to being arbitrarily close. In topological terms, for a system with positive dissipation (such as an aircraft), a group of trajectories will always shrink asymptotically onto an attracting 'set' or space which occupies zero volume in the phase space, i.e. transient motion decays onto an 'attractor'.

(d) *Attractors*

An attractor is a stable solution ('steady state') which attracts transient trajectories in phase space onto it. An attractor may be

1. a fixed point in phase space (corresponding to an equilibrium);
2. a periodic (closed loop) orbit in phase space (corresponding to a limit-cycle-type oscillation);
3. a toroidal-type orbit in phase space (corresponding to quasi-periodic behaviour);
4. a strange/chaotic attractor with fractal geometric properties (corresponding to chaotic steady-state motion).

The first attractor type corresponds to a solution for which all the eigenvalues have negative real parts. The second attractor type corresponds to all Floquet multipliers being in the unit circle. An attracting set must satisfy three mathematical criteria to be called an attractor:

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1. it must be an invariant set: i.e. x is a point on an invariant set X if $f^t(x) \in X$ for all t;
2. it must attract;
3. it must be indecomposable.

(*e*) *Stable, unstable and centre manifolds (insets and outsets)*

The topological term 'manifold' may be loosely described as a well-behaved 'surface' (not discontinuous, for example). The term 'surface' may relate to any number of dimensions in phase space. A trajectory in phase space is therefore a manifold (assuming it is well behaved). Of particular importance in nonlinear dynamics are a special set of manifolds: the common feature of all three manifolds is that they are the only trajectories which intercept a solution. They are called the *stable, unstable and centre manifolds*. A stable manifold is also called an *inset*, and an unstable manifold is also called an *outset*. Engineers are familiar with linearization about a solution to obtain eigenvalues and eigenvectors. Say, for example, a two-dimensional attractor has two stable eigenvectors (figure 1). There will be two stable manifolds to which the stable eigenvectors will be tangent at the solution point. The eigenvector with the larger eigenvalue will correspond to the fastest dynamics, and trajectories will converge in this eigendirection first, onto the slow eigendirection. Stable and unstable manifolds are therefore an extension of the concept of an eigenvector beyond the locality of a solution (in which linearization is valid). A centre manifold is associated with an eigenvector with an eigenvalue with a zero real part.

The main importance of the stable and unstable manifolds is two-fold as follows.

1. They may define a *basin of attraction* (as explained later).
2. They may connect at a point other than the solution: such a connection is called either *homoclinic* if a manifold connects with itself, or *heteroclinic* if a manifold connects with another manifold. Such connections are responsible for *global bifurcations*.

(*f*) *Saddle and repellors*

A solution may not be stable, as may be pictured by a ball on a surface (see figure 2). The cases in figure 2 are straightforward: both solutions are equilibria but figure 2*a* is stable and 2*b* is unstable. In two dimensions a solution may have the property of figure 3.

In figure 3, the ball is 'stable' in one direction but 'unstable' in another. Clearly, however, the system is not 'robust': a small perturbation will move the ball away from the equilibrium solution and so the system is said to be unstable. For such an equilibrium point in two dimensions this would correspond to a system with one eigenvalue with real part positive and the other with real part negative. The eigenvectors give information about the direction of trajectories in the vicinity of the equilibrium point in phase/state space. A solution with some stable and some unstable eigenvalues may be called a 'saddle' solution. If a solution is unstable in all eigendirections it may be called a 'repellor'. For a nonlinear system all four attractor types (equilibrium, periodic, quasi-periodic and chaotic) may exist in either stable/saddle/repellor forms.

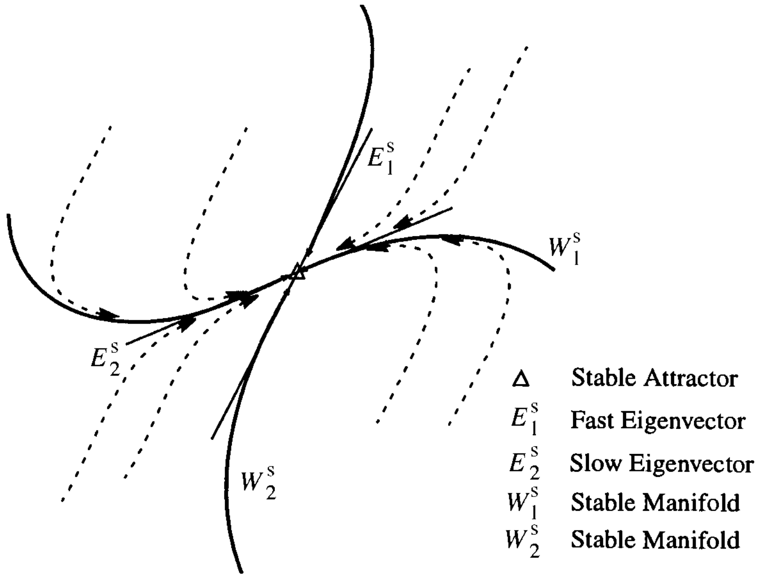

Figure 1. Stable manifolds, eigenvectors and trajectories for an attracting node.

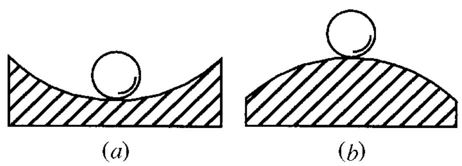

Figure 2. Two equilibria: (a) stable and (b) unstable.

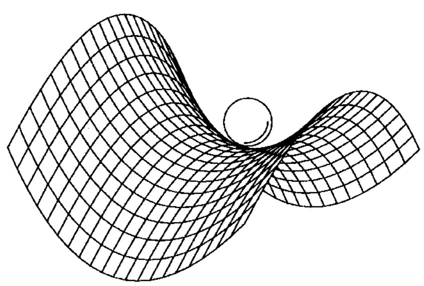

Figure 3. A saddle equilibrium.

(g) *Basins of attraction*

Each stable attractor attracts trajectories in its vicinity in phase space. The region within which trajectories are attracted is called the 'basin of attraction'. Basins of attraction define the limits of stability for a given attractor. An aircraft in a steady-state condition (a steady spin for example) must be perturbed until the aircraft's state exists outside the basin of attraction for the spin before the aircraft can change steady-state (say to normal flight). A nonlinear system may have more than one attractor co-existing in phase space for the same parameter setting: the boundaries between these competing attractors is usually smooth but may be fractal. The bound-

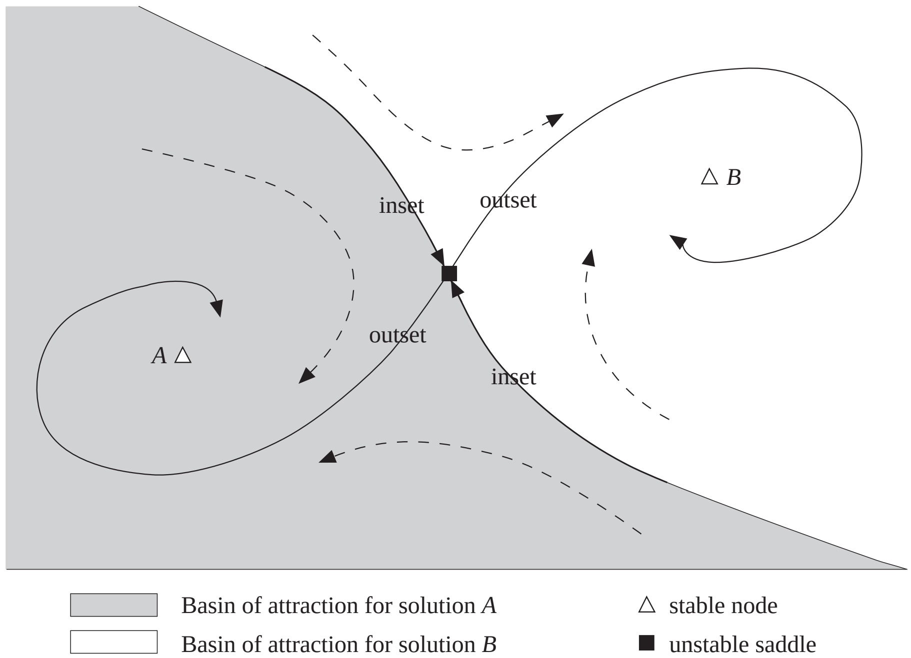

Figure 4. Basins of attraction boundaries defined by saddle-node insets.

aries are often described by the flow into unstable solutions of the system as in figure 4. That the trajectory into the saddle node defines the basin boundary is one reason for determining both stable and unstable solutions of a system.

(*h*) *Structural stability*

Stability of a solution, as is normally understood, relates to the behaviour of trajectories close to the solution (i.e. the solution with some perturbation). Structural stability of a system refers to the behaviour of 'nearby systems' (i.e. the system with some perturbation).

A *system* is said to be structurally stable if a small change in an *external* parameter does not *qualitatively* change the phase portrait (and therefore the behaviour of the system). Such external parameters may be noise or some disturbance: for example, a gust. A connection may be seen, therefore, between structural stability and robustness.

A *phase portrait* of a system may also be said to be structurally stable. However, in this case, structural stability means that the phase portrait does not change qualitatively if a control *parameter of the system* is changed (such as an aircraft's aerodynamic controls).

(*i*) *Bifurcation*

A bifurcation is a qualitative change in the systems dynamics as a control parameter is varied. Associated with this change will be a change in the topology of the attractor-basin phase portrait. At a point of bifurcation therefore, the phase portrait is structurally unstable. Bifurcations themselves may be structurally stable or unstable. The 'co-dimension' of a bifurcation is the number of control parameters required to make the bifurcation itself structurally stable. In the real world of imperfections, only structurally stable systems and bifurcations can exist. Bifurcations are also classed in other ways as follows.

1. *Local bifurcations* involve the creation, destruction or splitting of attractors (solutions). Local bifurcations occur when the eigenvalues of a system pass through the imaginary axis (or, for periodic solutions, when the Floquet multipliers pass through the unit circle). Such bifurcations may be associated with gain or loss of static or dynamic stability (i.e. stiffness or damping changes sign).

2. *Global bifurcations* typically involve connections between the insets and outsets of saddles.

The effect of a bifurcation may be a smooth change in behaviour or may result in a sudden jump to a new behaviour. The former is referred to as 'subtle', the latter as 'dangerous'. The outcome from the jump phenomena of a 'dangerous' bifurcation may be either determinate or indeterminate.

Many examples of different bifurcations in flight dynamics are given in the papers which follow. However, the two most common are the *saddle-node fold bifurcation* and the *supercritical Hopf bifurcation*. A saddle-node bifurcation is one of several bifurcations which may occur as a single real eigenvalue crosses the imaginary axis. As a control parameter is changed, at a saddle-node bifurcation, a stable node and unstable saddle solution meet and are both destroyed. This results in a sudden loss of any local solution (stable or unstable), hence the saddle-node is a dangerous bifurcation.

A *supercritical Hopf bifurcation* is one of two possible bifurcations which occur when a complex pair of eigenvalues cross the imaginary axis. In the supercritical case, the local equilibrium solution becomes unstable but a stable periodic solution develops. There is therefore no sudden loss of a stable local solution at a supercritical Hopf: it is a 'subtle' bifurcation. The *subcritical form* of the Hopf bifurcation also occurs as a complex pair of eigenvalues cross the imaginary axis. In the subcritical case an unstable periodic solution combines with an oscillatory stable equilibrium solution to give an oscillatory unstable equilibrium solution.

(*j*) *Poincaré section and map*

A Poincaré section is a means of analysing a solution with periodic/recurrent behaviour. It consists of taking a section ($\Sigma \in \mathbb{R}^{n-1}$) through phase space, transverse to the flow ($\Gamma \in \mathbb{R}^n$).

The Poincaré map is a discrete difference equation which maps points of intersection on the Poincaré section of the flow ($\Gamma \cap \Sigma$) to the next point of intersection. Hence a Poincaré map is sometimes called a first return map. The importance of Poincaré maps is that they retain important features of the flow; for example, stability is preserved. The map may be easier to analyse, however, because it exists in one less dimension and is discrete (which lends itself to numerical methods). The disadvantages of Poincaré maps are that the Poincaré section must be carefully defined, and the Poincaré map may be difficult to compute. For periodic orbits (such as

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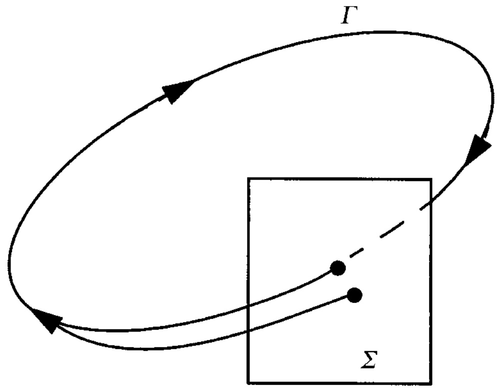

Figure 5. A Poincaré section (Σ) of a non-stationary solution (Γ).

wing-rock solutions analysed in several of the following papers), the stability and bifurcations of the orbit may be analysed either via a Poincaré map or via Floquet (small-perturbation analysis) theory.

3. Background to aircraft dynamics and modelling

For a thorough introduction to aircraft flight dynamics there are many good textbooks (see, for example, Babister 1961; Etkin 1958; Etkin & Reid 1996; Nelson 1989). The aim of this is to expain some key ideas and terminology of flight dynamics to aid understanding of the papers which follow. Flight dynamics (or aircraft stability and control) is concerned with the overall dynamic behaviour of aircraft: stability, controllability, dynamic response, handling qualities, etc. Flight-dynamics analysis therefore requires a comprehensive model of the aircraft. This model must be valid for all combinations of angle-of-attack, Mach number, 'g' and altitude in which the aircraft operates. This operational 'space' is referred to as the aircraft's 'flight envelope'. At the heart of an aircraft model are the rigid-body equations of motion. Considering an aircraft to be rigid, an aircraft has six degrees of freedom, giving a 12-state dynamic problem. Four of these states (the spatial position of the aircraft and its heading angle) do not effect the dynamic behaviour of interest. The remaining eight phase-space variables are usually denoted as in table 1.

Flight dynamicists distinguish between several axis systems and this can lead to several different possible combinations of state variables. The eight systems given in table 1 are the most frequently used. A common alternative is for the variables V, α and β to be replaced with three components of velocity (u, v and w) aligned with the body axes, where

$$\alpha = a\tan(w/u), \qquad \beta = a\sin(v/V), \qquad V = \sqrt{(u^2+v^2+w^2)}.$$

The rigid-body equations of motion in terms of body-axis states are

$$m\dot{u} - m_y\dot{r} + m_z\dot{q} = X + mrv - mqw + m_x(q^2+r^2) - m_ypq - m_zpr,$$

$$m\dot{v} - m_z\dot{p} + m_x\dot{r} = Y + mpw - mru + m_y(r^2+p^2) - m_zqr - m_xpq,$$

$$m\dot{w} - m_x\dot{q} + m_y\dot{p} = Z + mqu - mpv + m_z(p^2+q^2) - m_xpr - m_yqr,$$

Table 1. *Phase-space variables*

three rotational rates	
p	roll rate (about the longitudinal axis)
q	pitch rate (about the transverse or lateral axis)
r	yaw rate (about the directional or vertical axis)
three states relating the aircraft to the velocity vector	
V	total aircraft velocity
α	angle-of-attack: angle between the symmetric (longitudinal) component of velocity and the aircraft's longitudinal axis
β	angle of sideslip: angle between the plane of symmetry and the velocity vector
two Euler angles describing the aircraft orientation relative to the gravity vector	
θ	pitch attitude
ϕ	roll attitude (bank angle)

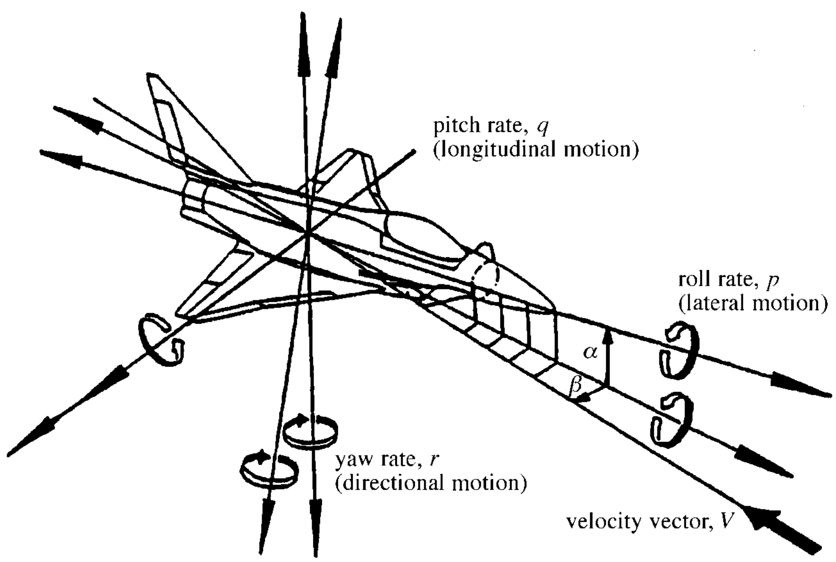

Figure 6. Aircraft axes and notation.

$$
\begin{aligned}
m_y\dot{w} - m_z\dot{v} + I_{xx}\dot{p} - I_{xz}\dot{r} - I_{xy}\dot{q} &= L - m_y(pv - qu) + m_z(ru - pw) \\
&\quad + (I_{yy} - I_{zz})qr - I_{yz}(r^2 - q^2) + I_{yx}pq - I_{xy}pr, \\
m_z\dot{u} - m_x\dot{w} + I_{yy}\dot{q} - I_{xy}\dot{p} - I_{yz}\dot{r} &= M - m_z(qw - rv) + m_x(pv - qu) \\
&\quad + (I_{zz} - I_{xx})pr - I_{xz}(p^2 - r^2) + I_{xy}qr - I_{yz}qp, \\
m_x\dot{v} - m_y\dot{u} + I_{zz}\dot{r} - I_{yz}\dot{q} - I_{xz}\dot{p} &= N - m_x(ru - pw) + m_y(qw + rv) \\
&\quad + (I_{xx} - I_{yy})pq - I_{xy}(q^2 - p^2) + I_{yz}pr - I_{xz}rq.
\end{aligned}
$$

In addition, there are two kinematic equations which relate the aircraft orientation to the gravity vector. If Euler angles are used (as opposed to direction cosines or quarternions), then the following equations suffice as long as rotations are applied in the order ψ, θ, ϕ:

$$\dot{\theta} = q\cos\phi - r\sin\theta,$$
$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta.$$

X, Y, Z and L, M, N are the aerodynamic and gravitational forces and moments acting on the aircraft at a fixed point, called the moment reference centre (MRC). m_x, m_y and m_z are the first mass moments relating the centre of gravity (CG) position to the MRC. Either the aerodynamic forces and moments may be referenced to the MRC and the above equations are used, or the aerodynamic forces and moments may be converted to act through the CG, in which case the mass moments may be set to zero. If the full equations of motion are required, they may be solved by using Gaussian elimination methods such as LU decomposition. Usually, however, simplification is common due to inertial symmetry leading to I_{xy} and I_{yz} being zero. However, asymmetric stores-configurations, for example, will lead to non-zero values of these cross-inertias. With these simplifications the equations of motion become

$$\dot{u} = \frac{X}{m} + rv - qw, \qquad \dot{v} = \frac{Y}{m} + pw - ru, \qquad \dot{w} = \frac{Z}{m} + qu - pv,$$

$$\dot{p} = \frac{L}{I_{xx}} + qr\frac{(I_{yy} - I_{zz})}{I_{xx}} - I_{xz}(pq + \dot{r}),$$

$$\dot{q} = \frac{M}{I_{yy}} + pr\frac{(I_{zz} - I_{xx})}{I_{yy}} + \frac{I_{xz}}{I_{yy}}(p^2 + r^2),$$

$$\dot{r} = \frac{N}{I_{zz}} + pq\frac{(I_{xx} - I_{yy})}{I_{zz}} - \frac{I_{xz}}{I_{zz}}(rp - \dot{p}).$$

Even with these simplifications the rigid-body equations of motion are nonlinear due to inertial effects. The most noticeable effect is 'roll coupling': because most combat aircraft are long and 'pointy', giving $I_{xx} \ll I_{yy}, I_{zz}$, a fast roll rate about the velocity vector (high p and r) leads, through the pitch rate equation, to a substantial pitch rate (q).

Furthermore, the equations include the total aerodynamic forces and moments acting on the aircraft. For conventional aircraft at low angles of attack and subcritical Mach numbers (typically $M < 0.7$), the aerodynamics may be relatively linear. For significant parts of the flight envelope of a high-performance aircraft, however, the aerodynamic forces and moments may be highly nonlinear. For this reason the aerodynamic forces and moments are typically modelled by use of multiple multi-dimensional interpolation tables. These tables are usually constructed from empirical data. While such a technique handles nonlinear trends of aerodynamic force and moment with flight condition, the technique does not easily account for nonlinearities such as hysteresis or multiple-valued functions. The importance of such nonlinearities is limited to a very small part of the flight envelope. Nevertheless, as nonlinear analysis is frequently applied to extreme parts of a flight envelope the adequacy of the model must always be borne in mind.

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Adding a control system adds to the number of states of the problem. In addition, a control system may result in degenerate solutions (an infinite number of solutions for the same control parameter setting). These problems are discussed by Avanzini & de Matteis (1996). Inclusion of aeroelastic effects in the model may also increase the number of state variables.

Usually, an aircraft will fly in a stable equilibrium state corresponding to steady level flight or a steady banked turn. In typical steady turns, values of α will be moderate (0–20°), values of β will be small and p, q and r will be steady. Other equilibrium states usually exist: a 'stall' in which the airflow over the wing is detached and turbulent is an equilibrium state characterized by large α (typically greater than 20°) and low speed. A 'spin' may also be an equilibrium state. A 'spin' is a multi-axis rotation of an aircraft with a stalled wing. Spins may exhibit various characteristics but are typically characterized by excessive yaw rates and moderate to very high angles of attack (30–90°). Bifurcation analysis has often been used to analyse spins; examples are given in the following papers.

In combat aircraft the most common periodic (limit-cycle) behaviour is 'wing rock'. This consists of a lateral oscillation with a period of about 2–3 s. The oscillation may develop at moderate angles of attack (before stall), and is an undesirable characteristic. Bifurcation analysis has also been used to analyse wing rock, and examples are given here in the papers by Liebst, Goman and Macmillen. Spins may also be oscillatory in nature.

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